Simplified Inapproximability of k-means

Melanie Schmidt

Joint work with Euiwoong Lee and John Wright

07.11.2015

Definition

The k-means problem

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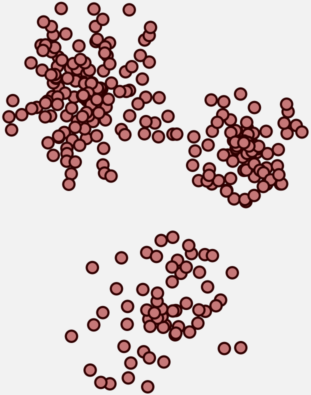

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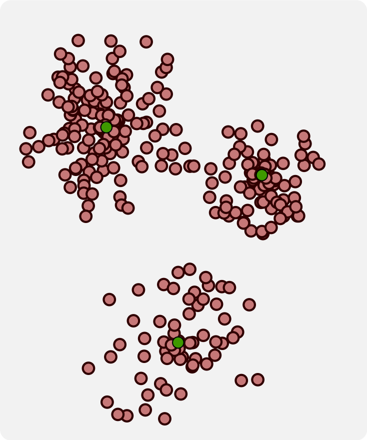

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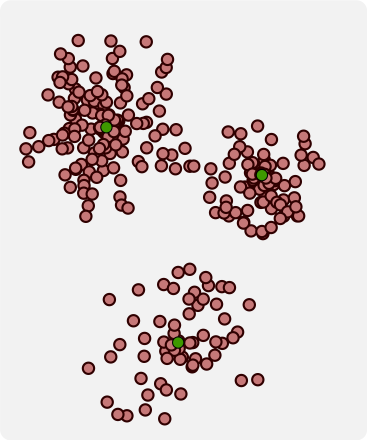

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the sum of the squared distances.

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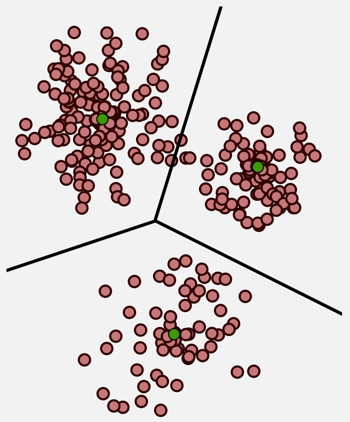

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- induces a partitioning of P

Complexity

	Small dimension d	Large dimension d
Small k		
Large k		

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[ADHP09] Aloise, Deshpande, Hansen, Popat: NP-hardness of Euclidean sum-of-squares clustering, Machine Learning, 2009.
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- What is the best possible approximation factor?
- PTAS for $d = 2$, constant d?

Reducing vertex cover ($\triangle$-free) to k-means

(Awasthi et. al., SoCG 2015)

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Vertex Cover instance

Graph $G = (V, E)$

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Vertex Cover instance

Graph $G = (V, E)$

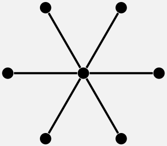
k-means instance

For $e = (i, j)$, define $x_e \in \mathbb{R}^{|V|}$ by

$$(x_e)_i = (x_e)_i = 1 \text{ and } (x_e)_\ell = 0 \text{ for } \ell \neq i, j$$

$$x_e = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0, \underbrace{1}_{j}, 0 \ldots, 0)$$

Example



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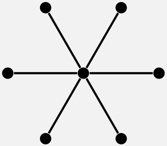

$$
\begin{array}{lcccccccccccc}
 & v_1 & v_2 & v_3 & \dots & & & & & & & & \\
e_1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
e_3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_3 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_3 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & \frac{1}{6} & 0 & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0
\end{array}
$$

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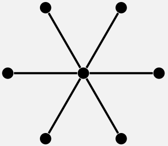

$$
\begin{array}{ll}
 & v_1 v_2 v_3 \ldots \\
e_1 & 101000000000 \\
e_2 & 001010000000 \\
e_3 & 001000000010 \\
e_3 & 001100000000 \\
e_3 & 001001000000 \\
e_3 & 001000100000 \\
 & \tfrac{1}{6}01\tfrac{1}{6}\tfrac{1}{6}\tfrac{1}{6}\tfrac{1}{6}000\tfrac{1}{6}0
\end{array}
$$

Cluster cost with (001000000000): $|E'|$

With centroid: $|E'| - 1$

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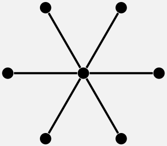

$v_1 v_2 v_3 \ldots$

e_1 101000000000
e_2 001010000000
e_3 001000000010
$\dot{e}_3$ 001100000000
e_3 001001000000
e_3 001000100000
$\frac{1}{6}\,0\,1\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,0\,0\,0\,\frac{1}{6}\,0$

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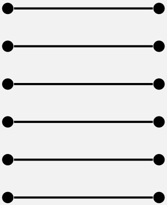

$v_1 v_2 v_3 \ldots$

e_1 101000000000
e_2 010000100000
e_3 000000011000
e_4 000000000101
e_5 000010000010
e_6 000101000000
$\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}\,\frac{1}{6}$

Cluster cost with (000000000000): $2|E'|$

With centroid: $2|E'| - 2$

Idea

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- small vertex cover implies k star clusters $\rightsquigarrow$ small cost $(m - k)$

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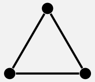
Problem: Triangles



$$\begin{array}{cccc} & v_1 & v_2 & v_3 \\ e_1 & 1 & 1 & 0 \\ e_2 & 0 & 1 & 1 \\ e_3 & 1 & 0 & 1 \\ & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{array}$$

Cluster cost:
$3 \cdot (2 \cdot \frac{1}{3^2} + (\frac{2}{3})^2) = 3 - 1$

Awasthi et. al., Part I

$(1+\varepsilon)$-hardness for vertex cover in $\triangle$-free graphs with $D \cdot n$ edges

$$\Downarrow$$

$(1+\varepsilon')$-hardness for k-means with $\varepsilon' \in \Theta(\varepsilon/D)$

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APX-hardness for VC in graphs with max. degree D

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APX-hardness for VC in $\triangle$-free graphs with max. degree poly(D, ε^{-1})

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VC in $\triangle$-free graphs is 1.36-hard

Awasthi et. al., Part I

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New Part II

APX-hardness for VC in graphs that are 4-regular

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APX-hardness for VC in $\triangle$-free graphs and maximum degree 4

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APX-hardness for VC in $\triangle$-free graphs and maximum degree 4

Chlebík, Clebíková, 2006

Given a 4-regular graph G, it is NP-hard to distinguish

- G has a vertex cover of size $\leq \alpha_{\text{min}}|V(A)|$
- every vertex cover in G has size $\geq \alpha_{\text{max}}|V(A)|$

Here, $\alpha_{\text{max}}/\alpha_{\text{min}} \geq 1.0192$.

Idea I

Replace every edge by three edges

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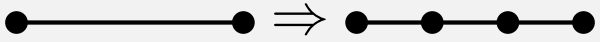

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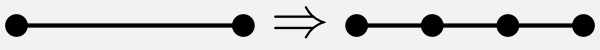
Replace every edge by three edges ($\rightsquigarrow +4n$ vertices and $+4n$ edges)



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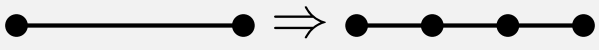

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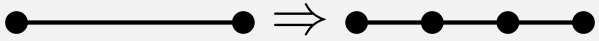

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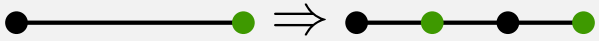

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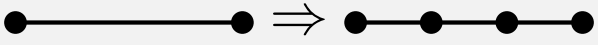

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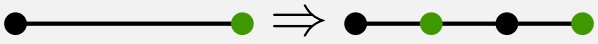

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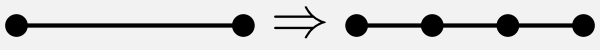

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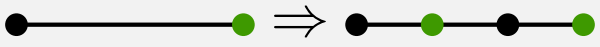
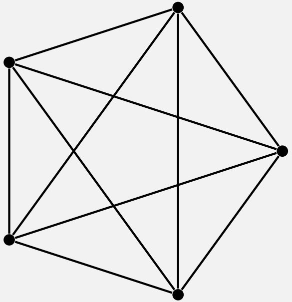


$\rightsquigarrow$ NP-hard to decide between $\leq (\alpha_{\text{min}} + 2)n$ and $\geq (\alpha_{\text{max}} + 2)n$

Idea II



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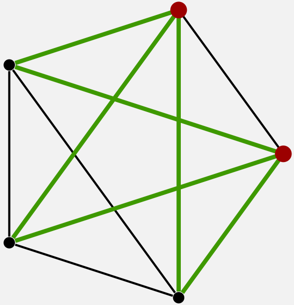

- Let E' with $|E'| \geq m/2$ be the edges of a large cut

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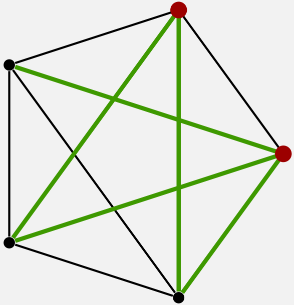

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- Pick $E_1 \subseteq E'$ with $|E_1| = m/2 = n$ (E_1 is bipartite)

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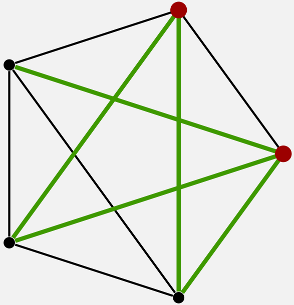

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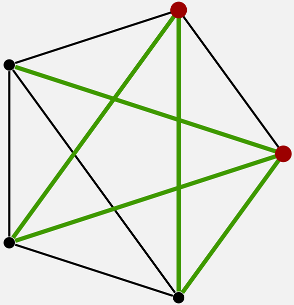

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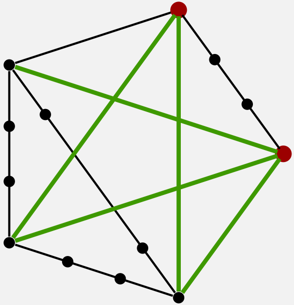

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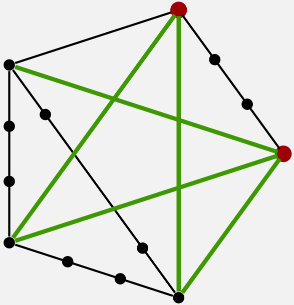

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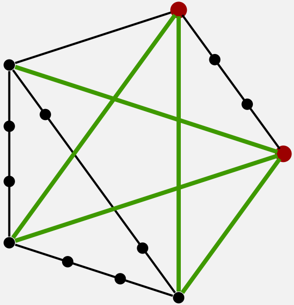
- Only split edges in E_2

$\rightsquigarrow$ $2n$ new edges and vertices, min. vertex cover size increases by n

Idea II



- Let E' with $|E'| \geq m/2$ be the edges of a large cut
- Pick $E_1 \subseteq E'$ with $|E_1| = m/2 = n$ (E_1 is bipartite)
- n remaining edges, E_2
- Only split edges in E_2

$\rightsquigarrow$ $2n$ new edges and vertices, min. vertex cover size increases by n
$\rightsquigarrow$ Gap between $(\alpha_{\min} + 1)n$ and $(\alpha_{\max} + 1)n$

Clebík, Chlebík, Clebíková, 2006

APX-hardness for VC in 4-regular graphs

Clebík, Chlebík, Clebíková, 2006

APX-hardness for VC in 4-regular graphs

New Part II

APX-hardness for VC in 4-regular graphs

$$\Downarrow$$

APX-hardness for VC in 4-regular $\triangle$-free graphs

Clebík, Chlebík, Clebíková, 2006

APX-hardness for VC in 4-regular graphs

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Awasthi et. al., Part I

$(1+\varepsilon)$-hardness for vertex cover in $\triangle$-free graphs with $D \cdot n$ edges

$$\Downarrow$$

$(1+\varepsilon')$-hardness for k-means with $\varepsilon' \in \Theta(\varepsilon/D)$

Clebík, Chlebík, Clebíková, 2006

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Theorem

It is NP-hard to approximate k-means within a factor of 1.0013.

Clebík, Chlebík, Clebíková, 2006

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Theorem

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Thanks!