A Local-Search Algorithm for Steiner Forest

M. Groß, A. Gupta, A. Kumar, J. Matuschke, D. Schmidt, J. Verschae
M. Schmidt

OR 2017, Berlin

September 8th 2011

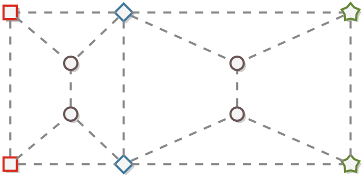
The Steiner Forest Problem

Input

Graph	$G = (V, E)$
Terminal pairs	$(s_1, \bar{s}_1), \ldots, (s_k, \bar{s}_k) \in V \times V$
Edge costs	$c : E \to \mathbb{R}^+$

Output

Minimum cost forest $F \subseteq E$ containing s_i-$\bar{s}_i$-path for all $i = 1, \ldots, k$



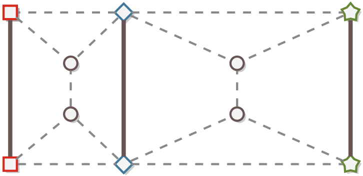
The Steiner Forest Problem

Input

Graph	$G = (V, E)$
Terminal pairs	$(s_1, \bar{s}_1), \ldots, (s_k, \bar{s}_k) \in V \times V$
Edge costs	$c : E \to \mathbb{R}^+$

Output

Minimum cost forest $F \subseteq E$ containing s_i-$\bar{s}_i$-path for all $i = 1, \ldots, k$



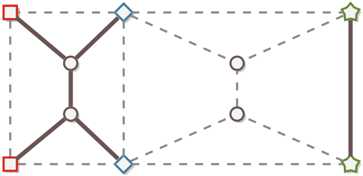
The Steiner Forest Problem

Input

Graph	$G = (V, E)$
Terminal pairs	$(s_1, \bar{s}_1), \ldots, (s_k, \bar{s}_k) \in V \times V$
Edge costs	$c : E \to \mathbb{R}^+$

Output

Minimum cost forest $F \subseteq E$ containing s_i-$\bar{s}_i$-path for all $i = 1, \ldots, k$



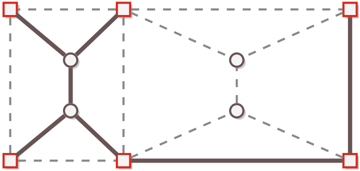
The Steiner Tree Problem

Input

Graph	$G = (V, E)$
Terminals	$s_1, \ldots, s_k \in V$
Edge costs	$c : E \to \mathbb{R}^+$

Output

Minimum cost tree $T \subseteq E$ containing all s_i



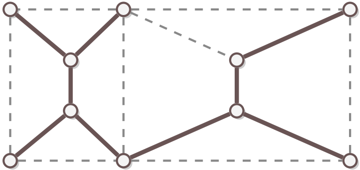
The Minimum Spanning Tree Problem

Input

Graph	$G = (V, E)$
Terminals	V
Edge costs	$c : E \to \mathbb{R}^+$

Output

Minimum cost tree $T \subseteq E$ containing all v



Known Simplifications

Known Simplifications

First

Use Metric Completion of $G \rightarrow$ no loss in the approximation guarantee

Known Simplifications

First

Use Metric Completion of $G \rightarrow$ no loss in the approximation guarantee

Second

Ignore Steiner nodes $\rightarrow$ factor 2 in the approximation guarantee

Known Simplifications

First

Use Metric Completion of $G \rightarrow$ no loss in the approximation guarantee

Second

Ignore Steiner nodes $\rightarrow$ factor 2 in the approximation guarantee

Simple 2-approximation for Steiner Tree

Compute MST on the metric completion of G

Known Simplifications

First

Use Metric Completion of $G \rightarrow$ no loss in the approximation guarantee

Second

Ignore Steiner nodes $\rightarrow$ factor 2 in the approximation guarantee

Simple 2-approximation for Steiner Tree

Compute MST on the metric completion of G

Example: Use Kruskal $\rightarrow$ Greedy 2-approximation for Steiner Tree

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual		
LP Rounding		
involved combinatorial		
Greedy / Gluttonous		
Local Search		

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual		
LP Rounding		
involved combinatorial		
Greedy / Gluttonous	2	
Local Search		

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual		
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	
Local Search		

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	
Local Search		

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search		

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	?

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	69 [this work]

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	69 [this work]

Why local search?

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	69 [this work]

Why local search?

- powerful technique, often used in practice, hope for new insights

Known Results

	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	69 [this work]

Why local search?

- powerful technique, often used in practice, hope for new insights
- dynamic Steiner Forest problem (no constant approximation known)

Known Results

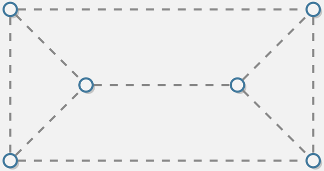
	Steiner Tree	Steiner Forest
Primal-Dual	$2 - \frac{2}{n}$	$2 - \frac{2}{n}$ [AKR95,GW95]
LP Rounding	1.39 [BGRS13]	
involved combinatorial	1.55 [RS05]	
Greedy / Gluttonous	2	96 [GK15]
Local Search	2	69 [this work]

Why local search?

- powerful technique, often used in practice, hope for new insights
- dynamic Steiner Forest problem (no constant approximation known)
- survivable network design problem (no combinatorial algorithm known)

Local Search

Example: Local search for metric MST

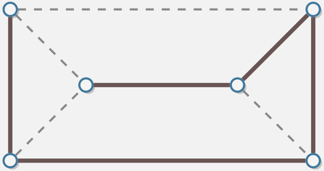


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

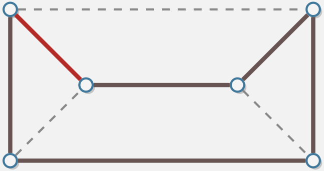


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

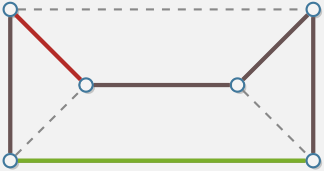


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

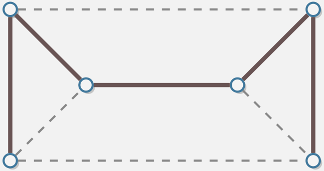


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

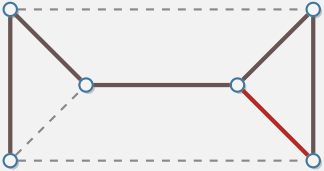


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

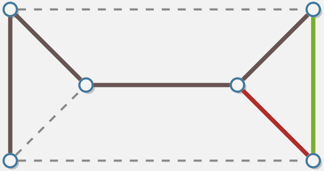


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

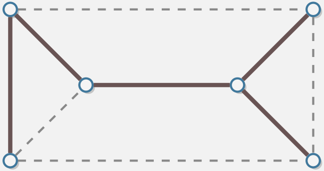


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

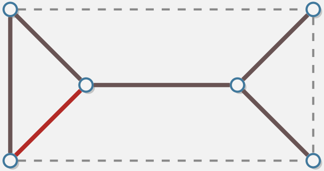


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

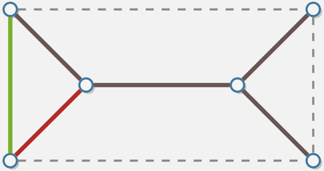


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

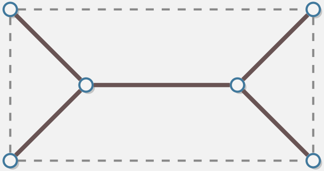


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

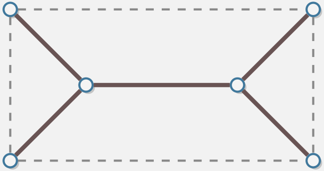


Local search algorithm.

1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

Local Search

Example: Local search for metric MST

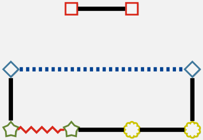


Local search algorithm.

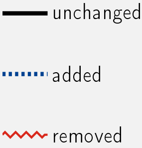
1. Start from arbitrary feasible solution.
2. Reach next feasible solution by executing single edge swaps.
3. Iterate until no improving swap $\rightsquigarrow$ Local optimum reached.

- For MST, this is optimal!
- $\rightsquigarrow$ 2-approximation for Steiner Tree

Building a local search algorithm: Moves

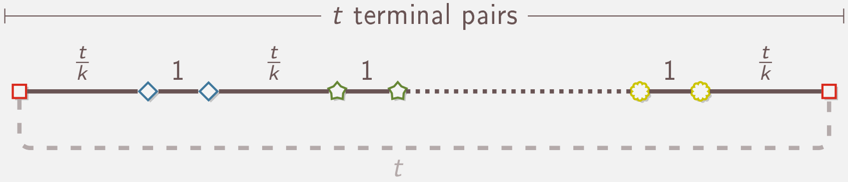


edge/edge swap



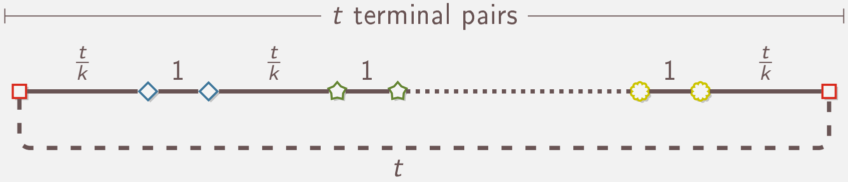
Local Search for Steiner Forest: Simple moves don't work

Choose parameter $k \in \omega(1)$ and number of terminals $t \in \omega(k)$



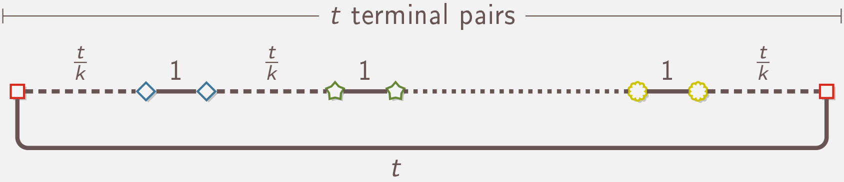
Local Search for Steiner Forest: Simple moves don't work

Choose parameter $k \in \omega(1)$ and number of terminals $t \in \omega(k)$



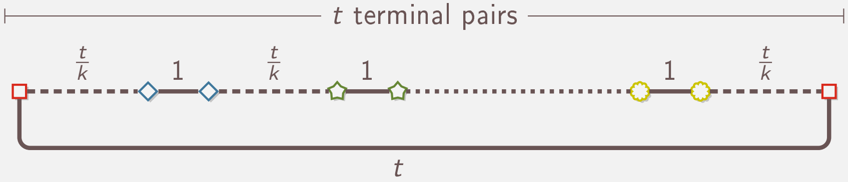
Local Search for Steiner Forest: Simple moves don't work

Choose parameter $k \in \omega(1)$ and number of terminals $t \in \omega(k)$



Local Search for Steiner Forest: Simple moves don't work

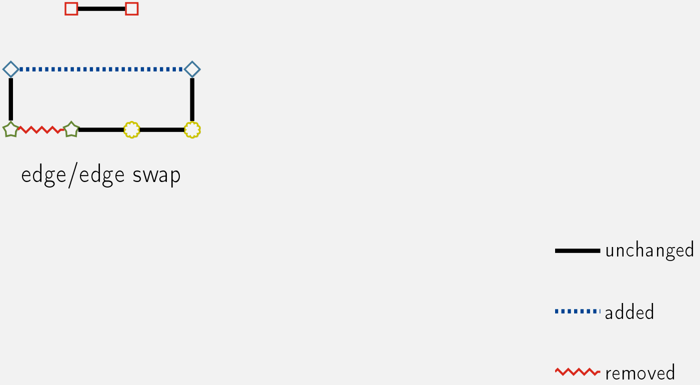
Choose parameter $k \in \omega(1)$ and number of terminals $t \in \omega(k)$



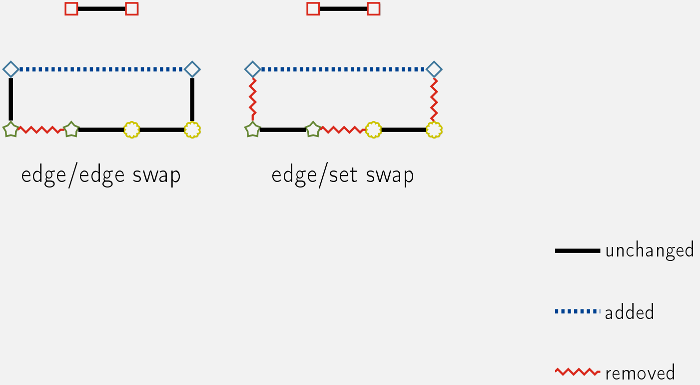
Need to remove $\omega(1)$ edges.

- Adding the long edge costs t
- Removing all short edges gains t^2/k, removing $< k$ edges gains $< t$
- Unless $> k$ edges are removed, no improving move, i.e., local optimum
- Local OPT $> t^2/k$ vs. global OPT $< 2t$

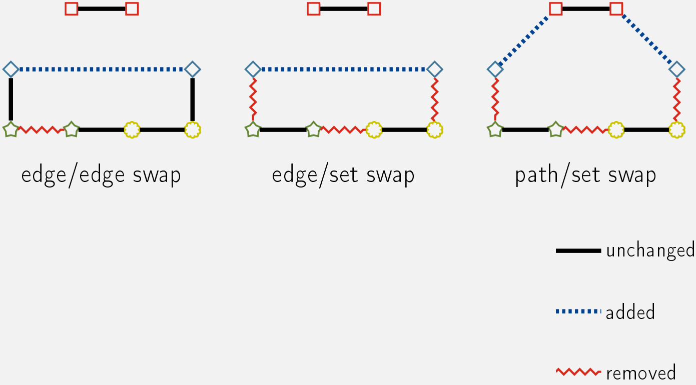
Building a local search algorithm: Moves



Building a local search algorithm: Moves



Building a local search algorithm: Moves



Local Search Algorithm for Steiner Forest: First Try

Local Search Algorithm for Steiner Forest: First Try

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)

Local Search Algorithm for Steiner Forest: First Try

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)

Local Search Algorithm for Steiner Forest: First Try

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves until local optimum

Local Search Algorithm for Steiner Forest: First Try

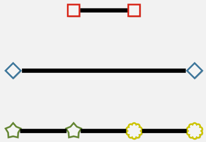
- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves until local optimum
- Drop inessential edges

Local Search Algorithm for Steiner Forest: First Try

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves until local optimum
- Drop inessential edges

Inessential edges

An edge is inessential if the solution is feasible without it.

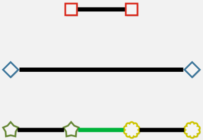


Local Search Algorithm for Steiner Forest: First Try

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves until local optimum
- Drop inessential edges

Inessential edges

An edge is inessential if the solution is feasible without it.



Spoiler

This does not work yet.

Spoiler

This does not work yet.

For now

Analysis of a case where it does work

Spoiler

This does not work yet.

For now

Analysis of a case where it does work

Setting

- solution $\mathcal{A}$
- optimum solution OPT
- $\mathcal{A}$ is locally optimal with respect to edge/set moves
- $\mathcal{A}$ is a tree!

For now: Local Optimum is a Tree

Strategy

- Charge edges of local opt $\mathcal{A}$ edges of global OPT.
- Recall: $\mathcal{A}$ is a tree, inessential edges dropped, no Steiner nodes

For now: Local Optimum is a Tree

Strategy

- Charge edges of local opt $\mathcal{A}$ edges of global OPT.
- Recall: $\mathcal{A}$ is a tree, inessential edges dropped, no Steiner nodes

Charging Argument

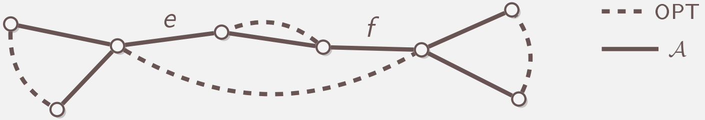
- Find assignment of $\mathcal{A}$-edges to OPT-edges such that no OPT-edge is assigned more than $\frac{7}{2}$ its cost.

$$\sum_{e \in \mathcal{A}} c(e) \leq \frac{7}{2} \sum_{e \in OPT} c(e)$$

- Assignment: Capacitated matching in a bipartite graph.

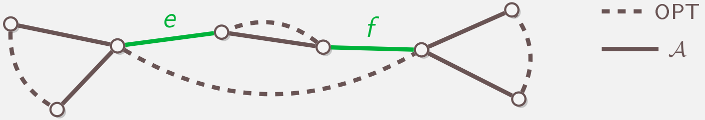
Compatible Edges

Edges e and f compatible w.r.t. OPT if...



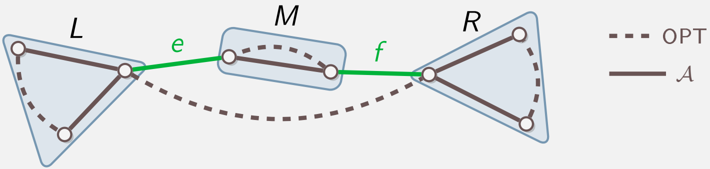
Compatible Edges

Edges e and f compatible w.r.t. OPT if...



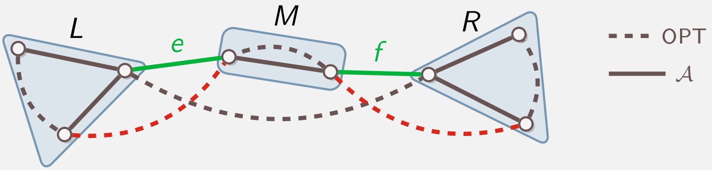
Compatible Edges

Edges e and f compatible w.r.t. OPT if...



Compatible Edges

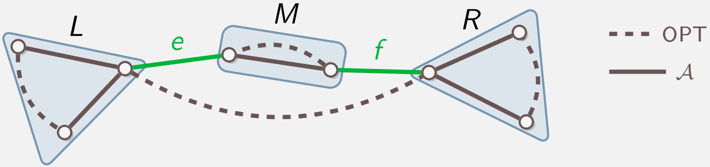
Edges e and f compatible w.r.t. OPT if...



No OPT-edges from L to M nor from M to R.

Compatible Edges

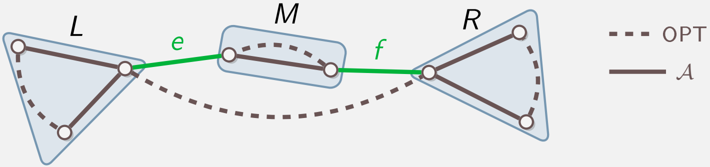
Edges e and f compatible w.r.t. OPT if...



No OPT-edges from L to M nor from M to R.

Compatible Edges

Edges e and f compatible w.r.t. OPT if...



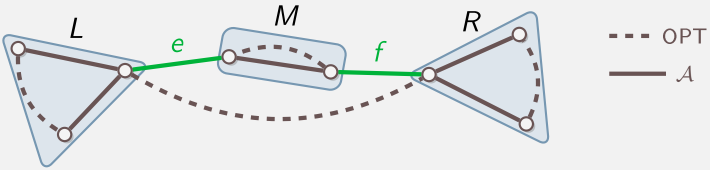
No OPT-edges from L to M nor from M to R.

Set of compatible edges behaves 'like one edge'

- Compatibility $\sim_{cp}$ is equivalence relation on edges
- Equivalence classes lie on paths
- Edge/Set swap can remove all edges of a class together

Compatible Edges

Edges e and f compatible w.r.t. OPT if...



No OPT-edges from L to M nor from M to R.

Set of compatible edges behaves 'like one edge'

- Compatibility $\sim_{cp}$ is equivalence relation on edges
- Equivalence classes lie on paths
- Edge/Set swap can remove all edges of a class together

Morally true: Inessential edges form one additional equivalence class

Charging Argument

Find assignment of $\mathcal{A}$-edges to OPT-edges such that no OPT-edge is assigned more than $\frac{7}{2}$ its cost.

$$\sum_{e \in \mathcal{A}} c(e) \leq \frac{7}{2} \sum_{e \in OPT} c(e)$$

Charging is done with a Hall-type argument; all edges of an equivalence class are charged together

Charging Argument

Find assignment of $\mathcal{A}$-edges to OPT-edges such that no OPT-edge is assigned more than $\frac{7}{2}$ its cost.

$$\sum_{e \in \mathcal{A}} c(e) \leq \frac{7}{2} \sum_{e \in OPT} c(e)$$

Charging is done with a Hall-type argument; all edges of an equivalence class are charged together

Recall

Any minimum spanning tree on the terminals is a 2-approximation for Steiner Tree.

Charging Argument

Find assignment of $\mathcal{A}$-edges to OPT-edges such that no OPT-edge is assigned more than $\frac{7}{2}$ its cost.

$$\sum_{e \in \mathcal{A}} c(e) \leq \frac{7}{2} \sum_{e \in OPT} c(e)$$

Charging is done with a Hall-type argument; all edges of an equivalence class are charged together

Recall

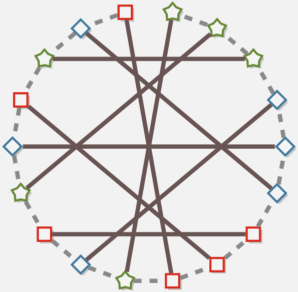
Any minimum spanning tree on the terminals is a 2-approximation for Steiner Tree.

Charging argument yields:

If a spanning tree on the terminals is locally optimal wrt edge/set swaps then it is a 7-approximation for Steiner Forest.

Local Search for Steiner Forest: Simple moves don't work

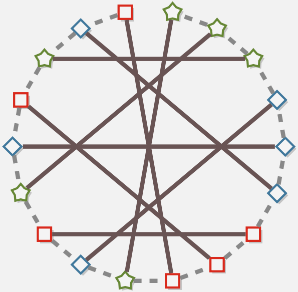
Another bad example



- $\mathcal{A}$: solid edges, OPT: dashed edges
- Solid edges cost 4, dashed edges cost 1

Local Search for Steiner Forest: Simple moves don't work

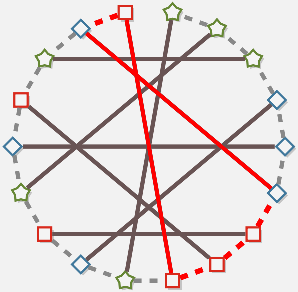
Another bad example



- $\mathcal{A}$: solid edges, OPT: dashed edges
- Solid edges cost 4, dashed edges cost 1
- Locally optimal for many simple swaps

Local Search for Steiner Forest: Simple moves don't work

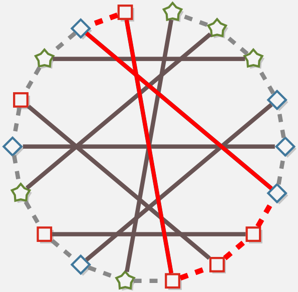
Another bad example



- $\mathcal{A}$: solid edges, OPT: dashed edges
- Solid edges cost 4, dashed edges cost 1
- Locally optimal for many simple swaps

Local Search for Steiner Forest: Simple moves don't work

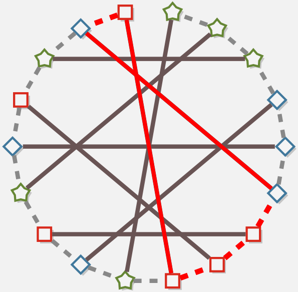
Another bad example



- $\mathcal{A}$: solid edges, OPT: dashed edges
- Solid edges cost 4, dashed edges cost 1
- Locally optimal for many simple swaps
- Factor is 36/17, but:
- More general version of this example gives $\Omega(\log n)$ lower bound

Local Search for Steiner Forest: Simple moves don't work

Another bad example



- $\mathcal{A}$: solid edges, OPT: dashed edges
- Solid edges cost 4, dashed edges cost 1
- Locally optimal for many simple swaps
- Factor is 36/17, but:
- More general version of this example gives $\Omega(\log n)$ lower bound
- Reason lies in high girth

Potential function

Potential function

- Need a move that connects components
- but just adding edges always increases the cost
- change objective for local search: potential function

Potential function

- Need a move that connects components
- but just adding edges always increases the cost
- change objective for local search: potential function

Width of a (sub)tree T:

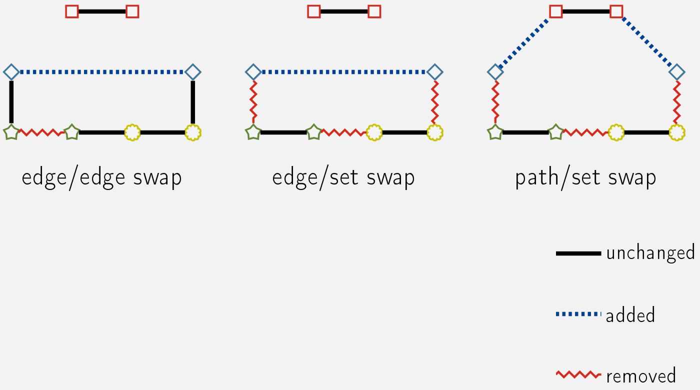
$$w(T) = \max_{\text{terminal pair } s_i, \bar{s}_i \text{ in } T} c(\{s_i, \bar{s}_i\})$$

Our potential adds the width:

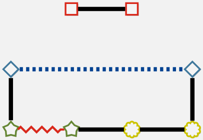
$$\phi(T) := w(T) + \sum_{e \in E(T)} c(e)$$

Forest: Add $\phi(T)$ for all subtrees T.

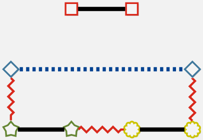
Improving moves



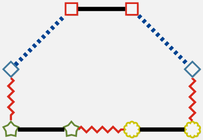
Improving moves



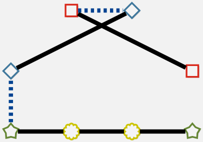
edge/edge swap



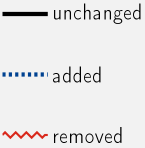
edge/set swap



path/set swap



connecting move



Local Search Algorithm for Steiner Forest: Final algorithm

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals
  (e.g., MST, direct connections)

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves and connecting moves until local optimum,

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves and connecting moves until local optimum, hereby evaluate solutions with respect to potential function

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves and connecting moves until local optimum, hereby evaluate solutions with respect to potential function
- Drop inessential edges

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves and connecting moves until local optimum, hereby evaluate solutions with respect to potential function
- Drop inessential edges

Theorem

There is a non-oblivious local search algorithm for the Steiner Forest Problem with a constant locality gap.

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves and connecting moves until local optimum, hereby evaluate solutions with respect to potential function
- Drop inessential edges

Theorem

There is a non-oblivious local search algorithm for the Steiner Forest Problem with a constant locality gap.

It can be implemented to run in polynomial time.

Local Search Algorithm for Steiner Forest: Final algorithm

- Ignore Steiner nodes ($\rightsquigarrow$ factor 2)
- Start with some feasible solution on terminals (e.g., MST, direct connections)
- Perform path/set moves and connecting moves until local optimum, hereby evaluate solutions with respect to potential function
- Drop inessential edges

Theorem

There is a non-oblivious local search algorithm for the Steiner Forest Problem with a constant locality gap.

It can be implemented to run in polynomial time.

Thank you for your attention!